Differential Geometry 2024, M.Math 2nd Year

Marks - 100, Duration - 3hrs

9th September, 2024

1. (10 points) Consider the regular curve
$$\alpha(t) = \left(t, \frac{t^2}{2}, \frac{t^3}{3}\right).$$
Compute its Frenet frame, the curvature and torsion when $t = 1$.

2. (10 points) Let I be an open interval around $0 \in \mathbb{R}$. Given $p \in \mathbb{R}^2$, a unit vector $u \in T_p\mathbb{R}^2$ and a smooth function $k(s) : I \to \mathbb{R}$, prove that there is at most one unit-speed curve $\alpha(s)$ in $\mathbb{R}^2$ such that
   (a) $\alpha(0) = p$,
   (b) $\alpha'(0) = u$
   (c) The signed curvature of $\alpha(s)$ is $k(s)$.

3. (10 points)(i) Define a map $F : \mathbb{RP}^2 \to \mathbb{R}^4$ by
$$F[x, y, z] = \frac{(x^2 - y^2, xy, xz, yz)}{x^2 + y^2 + z^2}.$$
Show that F is a smooth embedding.

4. ($5 + 10 = 15$ points)Let M be a compact connected manifold of dimension n and $f : M \to \mathbb{R}$ be a smooth function.
(i) Show that there exists a critical point of f. (a point $p \in M$ said to be a critical point of f if $df_p = 0$.)
(ii) Define a smooth function $f : \mathbb{RP}^n \to \mathbb{R}$ by
$$f([x_1, \cdots, x_{n+1}]) = \sum_{k=1}^{n+1} k x_k^2.$$
Show that the critical points of f are $u_1, \cdots, u_{n+1}$, where $u_i = [0, \cdots, 0, 1, 0, \cdots, 0]$, where the 1 occurs in the i th coordinate.

5. ($10 + 5 = 15$ points)Consider the map $F : \mathbb{R}^4 \to \mathbb{R}^2$ defined by
$$F(x, y, s, t) = (x^2 + y, x^2 + y^2 + s^2 + t^2 + y)$$
Show that $(0, 1)$ is a regular value of F, and that the regular level set $F^{-1}(0, 1)$ is diffeomorphic to $\mathbb{S}^2$.

6. (10 points) Let M be a smooth compact manifold. Show that there is no smooth submersion $F : M \to \mathbb{R}^k$ for any $k > 0$.

7. $(5 + 5{=}10$ points)Find integral curves of the following vector fields on $\mathbb{R}^2$

$$e^{-x}\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

   Is this vector field complete?
   (ii) Find a nowhere vanishing smooth vector field on $\mathbb{S}^{2n+1}$.

8. (10 points) Let M be a smooth manifold, and suppose f is a smooth function defined on a closed subset $A \subset M$. Show that for any open set U containing A, there exists a smooth function $\tilde{f} \in C^\infty(M)$ such that $\tilde{f}|_A = f|_A$ and $\mathrm{supp} f \subset U$ (f is smooth on A means there exists an open set W containing A such that $f : W \to \mathbb{R}$ is smooth).

9. (10 points) Let X be a vector field on M. If there exists a local one-parameter family $\{(U_\alpha, \phi_t^\alpha, \epsilon_\alpha)\}$ of local diffeomorphisms corresponding to X such that $\inf \epsilon_\alpha > 0$, then X is complete.